PHYS5010 — PLASMA PHYSICS

LECTURE 2 - PLASMA PROPERTIES: DENSITY, TEMPERATURE, AND DEBYE LENGTH

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Plasma properties: Density and Temperature.

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1 REVIEW: THERMODYNAMICS

Let us starting with a review of some important thermodynamical principles.

1.1 *First law of thermodynamics*

The *First Law of Thermodynamics* states that the change of the internal energy U is given by the sum of the work δW and heat δQ exchanged with the environment:

$$dU = \delta W + \delta Q \tag{1}$$

U is an extensive state function and a thermodynamical potential

Note the use of δ instead of d. This indicates that the amount of exchanged heat and work does depend on how the thermodynamical process is performed, and thus, δW and δQ are not *exact differentials*. In contrast, the change of the interior energy depends only on the initial and final state and is therefore an *exact differential*.

1.2 *Second law of thermodynamics*

The *Second Law of Thermodynamics* is closely related to the *entropy*, which is defined as the reversibly exchanged heat at constant temperature T

$$dS = \frac{\delta Q}{T}. \tag{2}$$

S is an extensive state function, while T is an intensive state function

The second law says now that for a closed system at equilibrium the entropy does not

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change, i.e.

$$dS = 0. \tag{3}$$

At a given temperature the amount of irreversibly exchanged heat is always smaller than the amount of reversibly exchanged heat, and thus

$$\delta Q_{irr} < \delta Q_{rev} = T\,dS. \tag{4}$$

For a closed system at equilibrium the entropy takes its maximum value S_{max}, while for an irreversible process $dS > 0$.

1.3 *Ideal gas*

In an ideal gas the particles are assumed to undergo only elastic collisions. In this case the equation of state is

$$pV = N\mathrm{k_B}T, \tag{5}$$

where p, V, and N are the pressure, volume, and particle number of the gas. The Boltzmann constant $\mathrm{k_B}$

$$\mathrm{k_B} = 1.308 \cdot 10{-}23\mathrm{J/K} = 8.617 \cdot 10^{-5} e\mathrm{V}. \tag{6}$$

relates the average kinetic energy of the gas with the temperature. For an ideal gas the average (translational) energy is

$$\frac{1}{2} m \langle\, v^2 \,\rangle = \frac{3}{2}\mathrm{k_B}T. \tag{7}$$

2 DENSITY

$u = 1.66 \cdot 10^{-27}$kg is the atomic mass unit.

SOLID As an example let us consider aluminum which has a density of $\rho_{Al} = 3 \cdot 10^3 \mathrm{kg/m^3}$ and an atomic mass of $m_{Al} = 27\mathrm{u}$. We now want to find the number of aluminum atoms per unit volume:

$$n_{Al} = \frac{\rho_{Al}}{m_{Al} u} = \frac{3 \cdot 10^3 \mathrm{kg/m^3}}{27 \cdot 1.66 \cdot 10^{-27}\mathrm{kg}} = 6.8 \cdot 10^{28} \mathrm{m^{-3}}. \tag{8}$$

AIR At standard pressure one mol of air has a volume of $22.4\mathrm{l} = 22.4 \cdot 10^{-3}\mathrm{m^3}$. One mol are $6 \cdot 10^{23}$ particles, and thus

$$n_{air} = \frac{6 \cdot 10^{23}}{22.4 \cdot 10^{-3}\mathrm{m^3}} = 2.7 \cdot 10^{25} \mathrm{m^{-3}}. \tag{9}$$

	$n[\mathrm{m}^{-3}]$	kT [eV]
Solar wind @ Earth	5	50
ionosphere	$10^5 - 10^6$	0.02
Solar corona	10^6	100
tokamak	10^{14}	10^4
laser-produced	10^{20}	100
glow discharge	$10^8 - 10^{10}$	2

3 TEMPERATURE

Let us have a closer look at the velocity distribution $f(\mathbf{v})$ of a gas and how it relates to its temperature. Because the gas motion is isotropic, $f(\mathbf{v})$ can only be a function of $\mathbf{v}^2$. On the other hand, the components of $f(\mathbf{v})$ must be independent, which implies that

$$f(\mathbf{v}^2) = f(v_x^2 + v_y^2 + v_z^2) = f(v_x^2) f(v_y^2) f(v_z^2). \tag{10}$$

The only function that fulfills Eq. (10) is

$$f(\mathbf{v}^2) = c \cdot e^{a\mathbf{v}^2}. \tag{11}$$

To find the constant c we require that the components of f are normalized, i.e. $\int f_i(v_i)\, dv = 1$, which is only possible if $a < 0$, and

$$1 = c \int e^{-av^2}\, dv = c\sqrt{\frac{\pi}{a}}. \tag{12}$$

To obtain the constant a we use that in a gas at equilibrium the energy per degree of freedom is $\frac{1}{2}\mathrm{k_B}T$, and therefore

$$\mathrm{k_B}T = m\langle\, v_i^2 \,\rangle = m \int v_i^2 f(v_i) d\, v_i = m\sqrt{\frac{\pi}{a}} \int \exp\left\{-av_i^2\right\} v_i^2 d\, v_i. \tag{13}$$

Replacing the argument of the exponential by $x = av_i^2$ we get

$dv_i = \frac{1}{2\sqrt{a}} \frac{dx}{\sqrt{x}}$

$$\mathrm{k_B}T = \frac{m}{\sqrt{\pi}a} \int_0^\infty e^{-x}\sqrt{x} dx = \frac{m}{\sqrt{\pi}a}\Gamma(\frac{3}{2}), \tag{14}$$

where the *Gamma function* $\Gamma(x)$ is defined as

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx \tag{15}$$

$$\Gamma(z+1) = \Gamma(z) \cdot z \tag{16}$$

$$\Gamma(1) = 1 \tag{17}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}. \tag{18}$$

From this follows that $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ and

$$f(v) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left\{-\frac{mv^2}{2k_B T}\right\}$$

$$f(\mathbf{v}) = \left\{\frac{m}{2\pi k_B T}\right\}^{3/2} \exp\left\{-\frac{m\mathbf{v}^2}{2k_B T}\right\}.$$

4 DEBYE SHIELDING

We now consider a *negative* test charge Q immersed in a homogeneous plasma. Q will attract ions but repell electrons. The displacement of electrons produces a *polarization charge*, which shields the plasma from the test charge. The theory of shielding has been developed first in 1923 by *Peter Debye* and *Erich Hückel* for dielectric fluids.

To derive the shielding potential ϕ for the charge Q we assume a homogeneous plasma with electrons of temperature T_e and density n_e and a fixed background of ions of density n_0. After the test charge has established equilibrium with the plasma its potential is given by the Poisson equation

$$\nabla^2 \phi(r) = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}(n_0 - n_e(r)) \text{ with } \phi(\infty) = 0. \tag{19}$$

In an electrostatic field the velocity distribution of the electrons is

$$f_e(\mathbf{v}) = n_0 \left\{\frac{m}{2\pi k_B T}\right\}^{3/2} \exp\left\{-\frac{\frac{1}{2}m\mathbf{v}^2 + q\phi(r)}{k_B T}\right\}.$$

The knowledge of $f_e(\mathbf{v})$ allows us to find the local electron number density $n_e(r)$

electrons: $q = -e$

$$n_e(r) = \int_{\mathbb{R}} f_e(\mathbf{v})\, d\mathbf{v} = n_0 \exp\left\{\frac{e\phi(r)}{k_B T}\right\},$$

which we substitute into Eq. (19)

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} n_0 \left(1 - \exp\left\{\frac{e\phi}{k_B T}\right\}\right).$$

We expand the exponential term into a Taylor series to linearize the quation for ϕ

$$\exp\left\{\frac{e\phi}{k_B T}\right\} = 1 + \frac{e\phi}{k_B T} + \frac{1}{2}\left(\frac{e\phi}{k_B T}\right)^2 + \frac{1}{3!}\left(\frac{e\phi}{k_B T}\right)^3 + \cdots$$

and keep only the first two terms

$$\nabla^2\phi \approx \frac{n_0}{\epsilon_0}\frac{e^2\phi}{k_B T}.$$

Because the plasma is isotropic we now want to make use of the spherical symmetry of the problem. To this aim we express the Laplace operator in spherical coordinates

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\phi\right) + \frac{1}{r^2\sin^2\theta}\partial^2_\phi\phi$$

and drop the symmetric angular terms

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) = \frac{n_0}{\epsilon_0}\frac{e^2\phi}{k_B T}.$$

This leads to an ordinary second order linear differential equation

$$\begin{aligned}
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) - \frac{n_0}{\epsilon_0}\frac{e^2\phi}{k_B T} &= 0\\
\frac{1}{r}\partial^2_r\left(r\phi\right) - \frac{n_0}{\epsilon_0}\frac{e^2\phi}{k_B T} &= 0\\
\partial^2_r\left(r\phi\right) - \frac{n_0}{\epsilon_0}\frac{e^2\phi}{k_B T}\left(r\phi\right) = y'' - \frac{n_0}{\epsilon_0}\frac{e^2\phi}{k_B T}y &= 0 \text{ with } y = \left(r\phi\right).
\end{aligned}$$

The solutions of $y'' + a^2 y = 0$ have the general form

$$y(x) = \frac{c}{x}\exp\left(\pm ax\right),$$

from which follows that

$$\phi(r) = \frac{A}{r}\exp\left(-\frac{r}{\lambda_D}\right)$$

with

$$\boxed{\lambda_D^2 = \frac{\epsilon_0 k_B T_e}{n_0 e^2}} \tag{20}$$

being the *Debye length*. The value for the constant A can be found by using the fact that at large distances $\phi(r)$ must asymptotically approach *Coulomb's law* and we yield the so-called *Debye-Hückel potential*

$$\boxed{\phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\exp\left(-\frac{r}{\lambda_D}\right)} \tag{21}$$

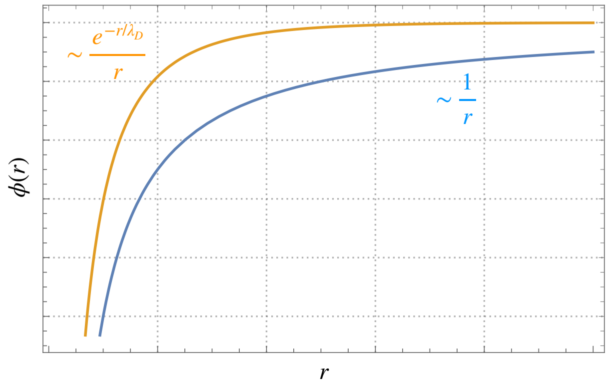

Figure 1: Comparison between the Debye-Hückel potential (orange) of a charge immersed in a plasma and the Coulomb potential (blue) of a free charge.

(Fig. 1). A useful relation for the Debye length is

$$\lambda_D = 7430\text{m}\sqrt{\frac{T}{e\text{V}}\frac{\text{m}^{-3}}{n}}. \tag{22}$$